ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2025, Vol. 86, No. 6, pp. 499–515.

Generalization of Gershgorin Circle Theorem with Application to Analysis and Design of Control Systems

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Received October 3, 2024
Revised February 28, 2025
Accepted March 14, 2025

Abstract—The application of the Gershgorin circle theorem and some of its derivatives to estimating matrix eigenvalues is considered. The obtained results are developed to design a localization region for matrix eigenvalues with interval-indefinite constant and non-stationary elements. The concept of *e-circles* is introduced to obtain more accurate estimates of these regions than when using Gershgorin circles. The obtained results are applied to the stability analysis of network systems, where it is shown that the proposed methods allows one to analyze a network with a much larger number of agents than when using methods for solving linear matrix inequalities in CVX and Yalmip/SeDuMi, as well as the eig (for calculating matrix eigenvalues) and lyap (for solving the Lyapunov equation) algorithms in MatLab. It is shown that if the developed methods are applied not to the system itself, but to the result obtained using the Lyapunov function method, then it is possible to study systems with matrices without diagonal dominance. This allowed us to consider the modification of the Demidovich condition for systems with non-stationary parameters and the design of the control law for non-stationary systems with matrices without diagonal dominance. All the obtained results are illustrated by numerical modeling.

Keywords: Gershgorin circle theorem, matrix eigenvalue localization domain, stability, control

DOI: 10.31857/S0005117925060012

1. INTRODUCTION

When analyzing the properties of dynamic systems and design of the control law, one of the key questions is whether the system is stable. Currently, various methods and approaches are used to determine stability: calculating the eigenvalues of the matrix [1], various algebraic and frequency stability criteria [1], the Lyapunov function method [1], divergent methods for studying stability [2], etc.

This paper focuses on design the localization domain of the eigenvalues of matrices with the application of the obtained result to the analysis and design of control laws. To construct the localization domain of the eigenvalues, the Gershgorin circle theorem [3–5] (hereinafter simply the Gershgorin theorem) and some of its consequences will be considered, and new results will be obtained on the generalization of this theorem to the case of parametrically indefinite matrices and matrices with non-stationary parameters.

Gershgorin theorem and its various modifications have been repeatedly considered in the literature. The interest in this theorem is associated with a simple method for determining the domain of localization of eigenvalues. Gershgorin theorem often leads to the study of systems containing matrices with diagonal dominance. In particular, such systems were studied in [6–8] and were called *super-stable* (if all Gershgorin circles are entirely in the left half-plane of the complex plane). It is shown that the analysis and design of control laws leads to linear programming problems. In [9–13], refinement of localization regions is obtained in the form of averaged estimates, using l_1 vector norms, etc., and in [14, 15], a design of static linear control laws using Gershgorin theorem is proposed. In [16–18], the application of Gershgorin theorem to the study of the stability of models in the chemical industry, models of electrical networks with three-phase generators, and biological models of epidemics is considered.

An analysis of the literature showed that when determining a localization region for the eigenvalues of matrices, Gershgorin theorem has advantages in the simplicity of its application, a convex procedure for finding the localization region, and low computational costs. However, limitations in the application of this theorem are associated with overestimated estimates of the localization region and consideration of matrices with diagonal dominance (or reduced to them using a diagonal matrix for transforming the basis). The requirement of diagonal dominance is also significantly restrictive in the design of the control law.

This paper will consider the solution of the following problems:

(1) estimates and localization regions of the eigenvalues of a constant matrix will be considered;

(2) localization regions of the eigenvalues of a matrix with interval parametric uncertainty will be obtained;

(3) the following will be considered as examples of application of the obtained results:

   (a) the problem of synchronization of network systems with a large number of scalar agents, where it will be shown that the proposed results can be applied to the stability analysis of a much larger number of agents than when using the methods for solving linear matrix inequalities in CVX and Yalmip/SeDuMi, as well as the eig (for calculating the eigenvalues of a matrix) and lyap (for solving the Lyapunov equation) algorithms in MatLab;

   (b) modification of the Demidovich condition (on the stability of linear systems with nonstationary parameters [19](Theorem 6.1), [23]) to systems with interval-uncertain nonstationary parameters and with the matrix of the original system without diagonal dominance;

   (c) the problem of finding a matrix in a linear control law using linear matrix inequalities for objects with a matrix without diagonal dominance.

The following *notations are used in the paper:* $\mathbb{C}$ is the set of complex numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space with the vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all real matrices of dimension $n \times m$ with the induced matrix norm $\|Q\|$, $\lambda_i\{Q\}$ is the ith eigenvalue of the square matrix Q, $\Re\{\lambda_i\{Q\}\}$ is the real part of the ith eigenvalue of the square matrix Q, $\Im\{\lambda_i\{Q\}\}$ is the imaginary part of the ith eigenvalue of the square matrix Q, I is the identity matrix of the corresponding order, the matrix $Q \in \mathbb{R}^{n\times m}$ will also be denoted as (q_{ij}), $i = 1, \ldots, n$, $j = 1, \ldots, m$.

2. ESTIMATES OF THE LOCALIZATION DOMAIN OF EIGENVALUES

2.1. Constant Matrices

In this section, estimates will be obtained for the localization region of the eigenvalues of the matrix $Q = (q_{ij}) \in \mathbb{R}^{n\times n}$ with constant elements. To clarify these estimates, the diagonal matrices $D = diag\{d_1, \ldots, d_n\}$ and $H = diag\{h_1, \ldots, h_n\}$ will be additionally considered. Let us introduce the notation of sums over rows and columns of absolute values of elements of matrices Q, $D^{-1}QD$

and $H^{-1}QH$ without diagonal elements in the form

$$R_i(Q) = \sum_{j=1,j\neq i}^{n} |q_{ij}|, \quad C_j(Q) = \sum_{i=1,i\neq j}^{n} |q_{ij}|,$$
$$R_i^D(Q) = \sum_{j=1,j\neq i}^{n} \frac{d_j}{d_i}|q_{ij}|, \quad C_j^D(Q) = \sum_{i=1,i\neq j}^{n} \frac{d_i}{d_j}|a_{ij}|,$$
$$R_i^H(Q) = \sum_{j=1,j\neq i}^{n} \frac{h_j}{h_i}|q_{ij}|, \quad C_j^H(Q) = \sum_{i=1,i\neq j}^{n} \frac{h_i}{h_j}|a_{ij}|.$$

Below are two lemmas that allow one to obtain lower and upper bounds on the real parts of the eigenvalues of the matrix Q.

Lemma 1. *Consider the matrix $Q \in \mathbb{R}^{n\times n}$. There exist $d_i > 0$, $h_i > 0$, $i = 1,\ldots,n$ such that the following estimates hold:*

$$\begin{aligned} &\max_i\{\Re\{\lambda_i\{Q\}\}\} \leqslant \sigma^D_{\max}\{Q\} \leqslant \sigma_{\max}\{Q\},\\ &\min_i\{\Re\{\lambda_i\{Q\}\}\} \geqslant \sigma^H_{\min}\{Q\} \geqslant \sigma_{\min}\{Q\}, \end{aligned} \tag{1}$$

where

$$\begin{aligned} \sigma_{\max}(Q) &= \min\Big\{\max_i\{q_{ii} + R_i(Q)\}, \max_j\{q_{jj} + C_j(Q)\}\Big\},\\ \sigma_{\min}(Q) &= \max\Big\{\min_i\{q_{ii} - R_i(Q)\}, \min_j\{q_{jj} - C_j(Q)\}\Big\},\\ \sigma^D_{\max}(Q) &= \min_D\Big\{\max_i\{q_{ii} + R^D_i(Q)\}, \max_j\{q_{jj} + C^D_j(Q)\}\Big\},\\ \sigma^H_{\min}(Q) &= \max_H\Big\{\min_i\{q_{ii} - R^H_i(Q)\}, \min_j\{q_{jj} - C^H_j(Q)\}\Big\}. \end{aligned} \tag{2}$$

Proof. By Gershgorin theorem [4] all eigenvalues of Q are contained in the union of n circles $\cup_{i=1}^n \left\{z \in \mathbb{C} : |z - q_{ii}| \leqslant \sum\limits_{j=1,j\neq i}^{n} |q_{ij}|\right\}$. Since Q^{T} has the same eigenvalues as Q, all eigenvalues of Q are also contained in the union of n circles $\cup_{i=1}^n \left\{z \in \mathbb{C} : |z - q_{ii}| \leqslant \sum\limits_{j=1,j\neq i}^{n} |q_{ji}|\right\}$. Therefore, $\sigma_{\min}\{Q\} \leqslant \min\limits_i\{\Re\{\lambda_i\{Q\}\}\}$ and $\sigma_{\max}\{Q\} \geqslant \max\limits_i\{\Re\{\lambda_i\{Q\}\}\}$.

Now consider the diagonal matrix $D = diag\{d_1,\ldots,d_n\}$. It is known that the eigenvalues of the matrices $D^{-1}QD$ and Q do not change. However, by varying the coefficients d_i, the radii of the Gershgorin circles for the matrix Q can be changed in the form $\cup_{i=1}^n \left\{z \in \mathbb{C} : |z - q_{ii}| \leqslant \sum\limits_{j=1,j\neq i}^{n} \frac{d_j}{d_i}|q_{ij}|\right\}$ and for the matrix Q^{T} in the form $\cup_{i=1}^n \left\{z \in \mathbb{C} : |z - q_{ii}| \leqslant \sum\limits_{j=1,j\neq i}^{n} \frac{d_i}{d_j}|q_{ji}|\right\}$. Therefore, there exist $d_i > 0$, $i = 1,\ldots,n$ such that $\sigma^D_{\max}\{Q\} \leqslant \sigma_{\max}\{Q\}$. However, it is impossible to simultaneously decrease the radii of all circles by varying d_i, i.e. when the radii of some circles decrease, the radii of others increase. Therefore, to obtain the estimate $\sigma^H_{\min}\{Q\} \geqslant \sigma_{\min}\{Q\}$, the matrix H is used instead of D.

Lemma 2. *Consider the matrix* $Q \in \mathbb{R}^{n\times n}$. *There exist* $d_i > 0$, $h_i > 0$, $i = 1,\dots,n$ *and* $\alpha, \beta \in [0,1]$ *such that the following estimates hold:*

$$\max_i\{\Re\{\lambda_i\{Q\}\}\} \leqslant \sigma^{D,\alpha}_{\max}\{Q\} \leqslant \sigma^{\alpha}_{\max}\{Q\},$$
$$\min_i\{\Re\{\lambda_i\{Q\}\}\} \geqslant \sigma^{H,\beta}_{\min}\{Q\} \geqslant \sigma^{\beta}_{\min}\{Q\}, \tag{3}$$

where

$$\sigma^{\alpha}_{\max}(Q) = \min_{\alpha}\left\{\max_i\{q_{ii} + [R_i(Q)]^{\alpha}[C_i(Q)]^{1-\alpha}\}\right\},$$
$$\sigma^{\beta}_{\min}(Q) = \max_{\beta}\left\{\min_i\{q_{ii} - [R_i(Q)]^{\beta}[C_i(Q)]^{1-\beta}\}\right\},$$
$$\sigma^{D,\alpha}_{\max}(Q) = \min_{D,\alpha}\left\{\max_i\{q_{ii} + [R^D_i(Q)]^{\alpha}[C^D_i(Q)]^{1-\alpha}\}\right\}, \tag{4}$$
$$\sigma^{H,\beta}_{\min}(Q) = \max_{H,\beta}\left\{\min_i\{q_{ii} - [R^H_i(Q)]^{\beta}[C^H_i(Q)]^{1-\beta}\}\right\}.$$

Proof. According to Ostrovsky theorem [4], all eigenvalues of the matrix Q are contained in the union of n circles $\cup_{i=1}^n\left\{z\in\mathbb{C} : |z-q_{ii}| \leqslant \left[\sum_{j=1,j\neq i}^{n}|q_{ij}|\right]^{\alpha}\left[\sum_{j=1,j\neq i}^{n}|q_{ji}|\right]^{1-\alpha}\right\}$. Therefore, there exists α such that $\sigma^{\alpha}_{\max}\{Q\} \geqslant \max_i\{\Re\{\lambda_i\{Q\}\}\}$. Analogously, we obtain that there exists β such that $\sigma^{\beta}_{\min}\{Q\} \leqslant \min_i\{\Re\{\lambda_i\{Q\}\}\}$.

By varying the coefficients d_i, the radii of the circles can be changed $\cup_{i=1}^n\left\{z\in\mathbb{C} : |z-q_{ii}| \leqslant \left[\sum_{j=1,j\neq i}^{n}\frac{d_j}{d_i}|q_{ij}|\right]^{\alpha}\left[\sum_{j=1,j\neq i}^{n}\frac{d_j}{d_i}|q_{ji}|\right]^{1-\alpha}\right\}$. Consequently, there exist d_i, $i = 1,\dots,n$ and α such that $\sigma^{D,\alpha}_{\max}\{Q\} \leqslant \sigma^{\alpha}_{\max}\{Q\}$. From similar reasoning it follows that there exist h_i, $i = 1,\dots,n$ and β such that $\sigma^{H,\beta}_{\min}\{Q\} \leqslant \sigma^{\beta}_{\max}\{Q\}$.

Corollary 1. *If any upper bound in Lemmas 1 and 2 takes a negative value, then it is an estimate of the degree of stability, the concept of which was introduced by Ya.Z. Tsypkin and P.V. Bromberg in* [20].

Corollary 2. *From the proofs of Lemmas 1 and 2 it also follows that by the intersection of the corresponding circles one can find the domain of localization of the eigenvalues of the matrix* Q, *from which one can find not only upper and lower bounds for the real parts of the eigenvalues, but also an upper bound for the imaginary part, which we denote as* $\hat{\Im}\{Q\} \geqslant \max_i\{\Im\{\lambda_i\{Q\}\}\}$. *The value of* $\hat{\Im}\{Q\}$ *is defined as the maximum value of the intersection of the circles along the imaginary axis. If the upper bound on the real part of the eigenvalue of* Q *is negative, then one can obtain an estimate of the oscillation* μ *in the form* $\mu \leqslant \hat{\mu} := \frac{\hat{\Im}\{Q\}}{|\max_i\{\Re\{\lambda_i\{Q\}\}\}|}$. *It is well known* [1] *that the oscillation is used to estimate the overshoot* Π *in the form* $\Pi \leqslant e^{\pi/\mu}$. *Then the new estimate of the degree of overshoot is defined as* $\Pi \leqslant e^{\pi/\hat{\mu}}$.

Knowing the estimate of the degree of stability, the estimate of oscillation and the lower estimate of the real part of the eigenvalue, we can construct a majorant and a minorant for the transient process of a linear system under a single step action, which is a development of the results of S.A. Chaplygin, N.N. Luzin, A.A. Feldbaum and A.M. Rubinchik [21, 22].

The Corollaries 1 and 2 will also be true in further generalizations of the obtained results to perturbed matrices. Let us demonstrate what was noted in the lemmas and corollaries using the following example.

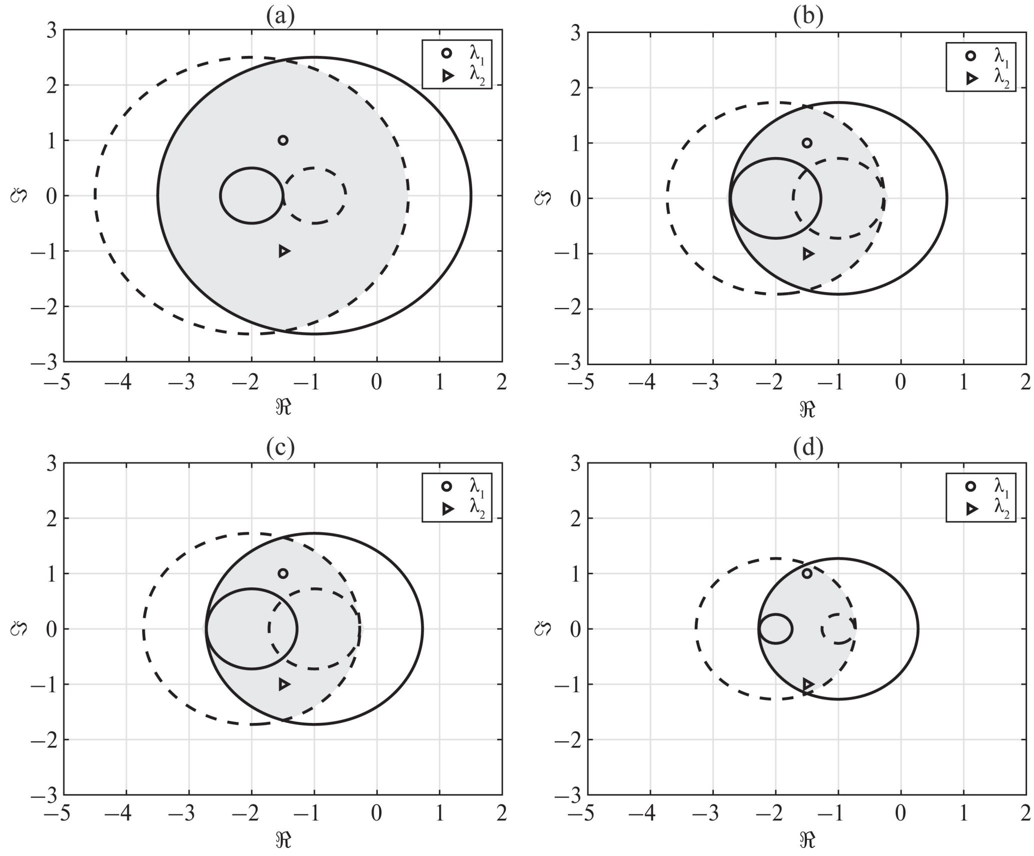

Fig. 1. Localization regions (gray region) of eigenvalues of matrix Q using estimates (1) and (3).

Example 1. Consider the matrix $Q = \begin{bmatrix} -1 & -2.5 \\ -0.5 & -2 \end{bmatrix}$, which eigenvalues are $-1.5 \pm i$. Figure 1 shows the localization regions (highlighted in gray) using:

- Gershgorin circle theorem (Fig. 1a);
- Lemma 1 with $H = diag\{1; 0.48\}$ and $D = diag\{1; 2.08\}$ (Fig. 1b);
- Lemma 2 with $\alpha = \beta = 0.23$ and $D = H = I$ (Fig. 1c);
- Lemma 2 with $\alpha = \beta = 0.01$, $H = diag\{1; 0.51\}$ and $D = diag\{1; 1.96\}$ (Fig. 1d);

The solid circles correspond to the circles which radii are calculated from the rows of the matrix, the dashed circles correspond to the columns of the matrix.

Table 1 contains upper and lower bounds for the real part of the eigenvalue of the matrix Q. The accuracy is calculated as the relative error between the corresponding estimate and $\Re\{\lambda\{Q\}\} = -1.5$ (e.g. $\frac{|-3.5+1.5|}{1.5} 100\% = 133.3\%$).

Table 1. Estimates of the real part of the eigenvalues of the matrix Q, obtained using (1) and (3)

Figure	Estimate of $\Re\{\lambda\{Q\}\}$	Accuracy, %
Fig. 1a	$\sigma_{\min}(Q) = -3.5$; $\sigma_{\max}(Q) = 0.5$	133.3
Fig. 1b	$\sigma^{H}_{\min}(Q) = -2.72$; $\sigma^{D}_{\max}(Q) = -0.27$	82
Fig. 1c	$\sigma^{\beta}_{\min}(Q) = -2.72$; $\sigma^{\alpha}_{\max}(Q) = -0.27$	82
Fig. 1d	$\sigma^{H,\beta}_{\min}(Q) = -2.26$; $\sigma^{D,\alpha}_{\max}(Q) = -0.73$	51.3

From Fig. 1 one can find estimates of the imaginary part, which are reflected in Table 2. The accuracy is calculated as the relative error between the corresponding estimate and $\Im\{\lambda\{Q\}\} = 1$.

Table 2. Estimates of the imaginary part of the eigenvalues obtained using Lemmas 1 and 2

Figure	Estimate of $\Im\{\lambda\{Q\}\}$	Accuracy, %
Fig. 1a	2.3	130
Fig. 1b	1.8	80
Fig. 1c	1.7	70
Fig. 1d	1.2	20

The best estimates of the real and imaginary parts are guaranteed by the result of Lemma 2, where the variable parameters D, H, α and β are used simultaneously.

2.2. Perturbed Matrices

In this section, we consider the search for the regions of localization of eigenvalues for matrices with interval-indefinite parameters:

$$\begin{aligned} &Q(t) = Q_0 + \Delta Q(t) \in \mathbb{R}^{n\times n}, && \\ &Q_0 = (q_{ij}^0), && \Delta Q(t) = (\Delta q_{ij}(t)), \\ &\Delta \underline{q}_{ii} \leqslant \Delta q_{ii}(t) \leqslant \Delta \overline{q}_{ii}, && |\Delta q_{ij}(t)| \leqslant m_{ij} \text{ for } i \neq j. \end{aligned} \tag{5}$$

Since the matrix elements can take any values from the admissible intervals, instead of the circles of localization of the eigenvalues considered in the proofs of Lemmas 1 and 2, we introduce the following figure into consideration.

Definition 1. The figure formed by the union of the circles $\mathcal{EC} = \cup_{q\in[\underline{q};\overline{q}]}\Big\{z \in \mathbb{C} : |z - q| \leqslant R\Big\}$ is called the e-circle.

We introduce notations for upper bounds of the sums over rows and columns of the absolute values of the elements of the matrices $Q(t)$, $D^{-1}Q(t)D$ and $H^{-1}Q(t)H$, excluding the diagonal elements, in the form

$$\begin{aligned} \hat{R}_i(Q) &= \sum_{j=1, j\neq i}^{n} (|q_{ij}^0| + m_{ij}), & \hat{C}_j(Q) &= \sum_{i=1, i\neq j}^{n} (|q_{ij}^0| + m_{ij}), \\ \hat{R}_i^D(Q) &= \sum_{j=1, j\neq i}^{n} \frac{d_j}{d_i}(|q_{ij}^0| + m_{ij}), & \hat{C}_j^D(Q) &= \sum_{i=1, i\neq j}^{n} \frac{d_i}{d_j}(|q_{ij}^0| + m_{ij}), \\ \hat{R}_i^H(Q) &= \sum_{j=1, j\neq i}^{n} \frac{h_j}{h_i}(|q_{ij}^0| + m_{ij}), & \hat{C}_j^H(Q) &= \sum_{i=1, i\neq j}^{n} \frac{h_i}{h_j}(|q_{ij}^0| + m_{ij}). \end{aligned}$$

Now we will consider the generalization of Lemmas 1 and 2 to the case of matrices with interval-indefinite elements. Below, in the formulations of the lemmas, we will omit the dependence of matrices and parameters on t for the sake of simplifying the expressions.

Lemma 3. *The eigenvalues of the matrix Q from* (5) *are in the intersection region of the e-circles*

$$\mathcal{EC}_{\text{row}} \cap \mathcal{EC}_{\text{col}}, \tag{6}$$

where

$$\begin{aligned} &\mathcal{EC}_{\text{row}} = \cup_{i=1}^{n} \mathcal{EC}_{\text{row},i}, \\ &\mathcal{EC}_{\text{row},i} = \cup_{\Delta q_{ii} \in [\Delta \underline{q}_{ii}; \Delta \overline{q}_{ii}]} \Big\{ \lambda \in \mathbb{C} : |\lambda - q_{ii}^0 - \Delta q_{ii}| \leqslant \hat{R}_i(Q) \Big\}, \end{aligned} \tag{7}$$

$$\begin{aligned} &\mathcal{EC}_{\text{col}} = \cup_{j=1}^{n} \mathcal{EC}_{\text{col},j}, \\ &\mathcal{EC}_{\text{col},j} = \cup_{\Delta q_{jj} \in [\Delta \underline{q}_{jj}; \Delta \overline{q}_{jj}]} \Big\{ \lambda \in \mathbb{C} : |\lambda - q_{jj}^0 - \Delta q_{jj}| \leqslant \hat{C}_j(Q) \Big\}. \end{aligned} \tag{8}$$

Proof. Let $\lambda(t)$ be an eigenvalue of the matrix $Q(t)$ and $s(t) = col\{s_1(t), \ldots, s_n(t)\}$ be the eigenvector corresponding to this eigenvalue. Choose the ith component of the vector $s(t)$ such that $\sup\{s_i(t)\} \geqslant \max\{\sup\{s_1(t)\}, \ldots, \sup\{s_{i-1}(t)\}, \sup\{s_{i+1}(t)\}, \ldots, \sup\{s_n(t)\}\}$. Denote $\bar{s}_i = \sup\{s_i(t)\}$. From the relation $\lambda(t)s(t) = Q(t)s(t)$ we write out the expression for the ith coordinate in the form $\lambda(t)s_i(t) = \sum\limits_{j=1}^{n} q_{ij}(t)s(t)$ or $(\lambda(t) - q_{ii}(t))s_i(t) = \sum\limits_{j=1, j\neq i}^{n} q_{ij}(t)s(t)$. Using the triangle inequality, we consider the estimate

$$\begin{aligned} |\lambda(t) - q_{ii}(t)||s_i(t)| &= \left| \sum_{j=1, j\neq i}^{n} q_{ij}(t)s_j(t) \right| \\ \leqslant \sum_{j=1, j\neq i}^{n} |q_{ij}(t)s_j(t)| &\leqslant \sum_{j=1, j\neq i}^{n} |q_{ij}(t)||s_j(t)| \leqslant \bar{s}_i \sum_{j=1, j\neq i}^{n} |q_{ij}(t)|. \end{aligned} \tag{9}$$

Let us rewrite the expression (9) as $|\lambda(t) - q_{ii}(t)||s_i(t)| - \bar{s}_i \sum\limits_{j=1, j\neq i}^{n} |q_{ij}(t)| \leqslant 0$ or in the form

$$\bar{s}_i \left(|\lambda(t) - q_{ii}(t)| \frac{|s_i(t)|}{\bar{s}_i} - \sum_{j=1, j\neq i}^{n} |q_{ij}(t)| \right) \leqslant 0. \tag{10}$$

Since $\frac{|s_i(t)|}{\bar{s}_i} \leqslant 1$, then the expression (10) will be satisfied if inequality holds

$$|\lambda(t) - q_{ii}(t)| \leqslant \sum_{j=1, j\neq i}^{n} |q_{ij}(t)|. \tag{11}$$

Since $\Delta \underline{q}_{ii} \leqslant \Delta q_{ii}(t) \leqslant \Delta \overline{q}_{ii}$ and $|\Delta q_{ij}(t)| \leqslant m_{ij}$ for $i \neq j$, then we rewrite the inequality (11) as an e-circle $\mathcal{EC}_{\text{row},i}$ from (7).

The relation (7) is satisfied for some i. Since it is unknown which i corresponds to a given $\lambda(t)$, we can only say that $\lambda(t)$ belongs to the union of e-circles $\mathcal{EC}_{\text{row}} = \cup_{i=1}^{n} \mathcal{EC}_{\text{row},i}$. This means that all eigenvalues of the matrix $Q(t)$ are contained in the union of e-circles $\mathcal{EC}_{\text{row}}$.

Since the matrix $Q^{\mathrm{T}}(t)$ has the same eigenvalues as the matrix $Q(t)$, then all eigenvalues of the matrix $Q(t)$ are contained in the union of e-circles $\mathcal{EC}_{\text{col}} = \cup_{j=1}^{n} \mathcal{EC}_{\text{col},j}$, see (8). Further reasoning for the matrix $Q^{\mathrm{T}}(t)$ is similar to that for the matrix $Q(t)$. Since the eigenvalues of the matrix $Q(t)$ are simultaneously in $\mathcal{EC}_{\text{row}}$ and $\mathcal{EC}_{\text{col}}$, they are in the domain (6).

Lemma 4. *Let $d_i > 0$, $h_i > 0$, $i = 1, \ldots, n$ be given. The eigenvalues of the matrix Q from* (5) *are in the intersection region of the e-circles*

$$\mathcal{EC}_{\text{row}}^{D} \cap \mathcal{EC}_{\text{col}}^{H},$$

where

$$\mathcal{EC}^D_{\text{row}} = \cup_{i=1}^n \mathcal{EC}^D_{\text{row},i},$$
$$\mathcal{EC}^D_{\text{row},i} = \cup_{\Delta q_{ii} \in [\Delta \underline{q}_{ii}; \Delta \overline{q}_{ii}]} \Big\{ \lambda \in \mathbb{C} : |\lambda - q^0_{ii} - \Delta q_{ii}| \leqslant \hat{R}^D_i(Q) \Big\},$$

$$\mathcal{EC}^H_{\text{col}} = \cup_{j=1}^n \mathcal{EC}^H_{\text{col},j},$$
$$\mathcal{EC}^H_{\text{col},j} = \cup_{\Delta q_{jj} \in [\Delta \underline{q}_{jj}; \Delta \overline{q}_{jj}]} \Big\{ \lambda \in \mathbb{C} : |\lambda - q^0_{jj} - \Delta q_{jj}| \leqslant \hat{C}^H_j(Q) \Big\}.$$

Proof. The results of Lemma 4 follow from Lemma 3 and the fact that the eigenvalues of the matrices $D^{-1}Q(t)D$, $H^{-1}Q(t)H$, and $Q(t)$ are the same.

Lemma 5. *Let $d_i > 0$, $i = 1, \ldots, n$ and $\alpha \in [0; 1]$ be given. The eigenvalues of the matrix Q from (5) are in the intersection region of the e-circles*

$$\mathcal{EC}^{D,\alpha} = \cup_{i=1}^n \mathcal{EC}^{D,\alpha}_i,$$

where

$$\mathcal{EC}^{D,\alpha}_i = \cup_{\Delta q_{ii} \in [\Delta \underline{q}_{ii}; \Delta \overline{q}_{ii}]} \Big\{ \lambda \in \mathbb{C} : |\lambda - q_{ii} - \Delta q_{ii}| \leqslant [\hat{R}^D_i(Q)]^\alpha [\hat{C}^D_i(Q)]^{1-\alpha} \Big\}.$$

Proof. The proof of Lemma 5 follows from the proofs of Lemmas 2 and 3 and the fact that the eigenvalues of the matrices $D^{-1}Q(t)D$ and $Q(t)$ are the same.

Corollary 3. *Similarly to Lemmas 1 and 2, one can write out estimates for the maximum and minimum values of the eigenvalues of the matrix Q using the results of Lemmas 3–5, i.e. there exist numbers $d_i > 0$, $h_i > 0$, $i = 1, \ldots, n$ and $\alpha, \beta \in [0; 1]$ such that the following estimates are valid:*

$$\begin{aligned}
&\max_i \Big\{ \sup_t \{\Re\{\lambda_i\{Q(t)\}\}\} \Big\} \leqslant \sigma^D_{\max}\{Q(t)\} \leqslant \sigma_{\max}\{Q(t)\},\\
&\min_i \Big\{ \sup_t \{\Re\{\lambda_i\{Q(t)\}\}\} \Big\} \geqslant \sigma^H_{\min}\{Q(t)\} \geqslant \sigma_{\min}\{Q(t)\},\\
&\max_i \Big\{ \sup_t \{\Re\{\lambda_i\{Q(t)\}\}\} \Big\} \leqslant \sigma^{D,\alpha}_{\max}\{Q(t)\} \leqslant \sigma^\alpha_{\max}\{Q(t)\},\\
&\min_i \Big\{ \sup_t \{\Re\{\lambda_i\{Q(t)\}\}\} \Big\} \geqslant \sigma^{H,\beta}_{\min}\{Q(t)\} \geqslant \sigma^\beta_{\min}\{Q(t)\},
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\sigma_{\max}(Q(t)) &= \min \Big\{ \max_i \{q^0_{ii} + \Delta \overline{q}_{ii} + \hat{R}_i(Q)\}, \max_j \{q^0_{jj} + \Delta \overline{q}_{jj} + \hat{C}_j(Q)\} \Big\},\\
\sigma_{\min}(Q(t)) &= \max \Big\{ \min_i \{q^0_{ii} - \Delta \overline{q}_{ii} - \hat{R}_i(Q)\}, \min_j \{q^0_{jj} - \Delta \overline{q}_{jj} - \hat{C}_j(Q)\} \Big\},\\
\sigma^D_{\max}(Q(t)) &= \min_D \Big\{ \max_i \{q^0_{ii} + \Delta \overline{q}_{ii} + \hat{R}^D_i(Q)\}, \max_j \{q^0_{jj} + \Delta \overline{q}_{jj} + \hat{C}^D_j(Q)\} \Big\},\\
\sigma^H_{\min}(Q(t)) &= \max_H \Big\{ \min_i \{q^0_{ii} - \Delta \overline{q}_{ii} - \hat{R}^H_i(Q)\}, \min_j \{q^0_{jj} - \Delta \overline{q}_{jj} - \hat{C}^H_j(Q)\} \Big\},
\end{aligned}$$

$$\begin{aligned}
\sigma^\alpha_{\max}(Q(t)) &= \min_\alpha \left\{ \max_i \{q^0_{ii} + \Delta \overline{q}_{ii} + [\hat{R}_i(Q)]^\alpha [\hat{C}_i(Q)]^{1-\alpha}\} \right\},\\
\sigma^\beta_{\min}(Q(t)) &= \max_\beta \left\{ \min_i \{q^0_{ii} - \Delta \overline{q}_{ii} - [\hat{R}_i(Q)]^\beta [\hat{C}_i(Q)]^{1-\beta}\} \right\},\\
\sigma^{D,\alpha}_{\max}(Q(t)) &= \min_{D,\alpha} \left\{ \max_i \{q^0_{ii} + \Delta \overline{q}_{ii} + [\hat{R}^D_i(Q)]^\alpha [\hat{C}^D_i(Q)]^{1-\alpha}\} \right\},\\
\sigma^{H,\beta}_{\min}(Q(t)) &= \max_{H,\beta} \left\{ \min_i \{q^0_{ii} - \Delta \overline{q}_{ii} - [\hat{R}^H_i(Q)]^\beta [\hat{C}^H_i(Q)]^{1-\beta}\} \right\}.
\end{aligned}$$

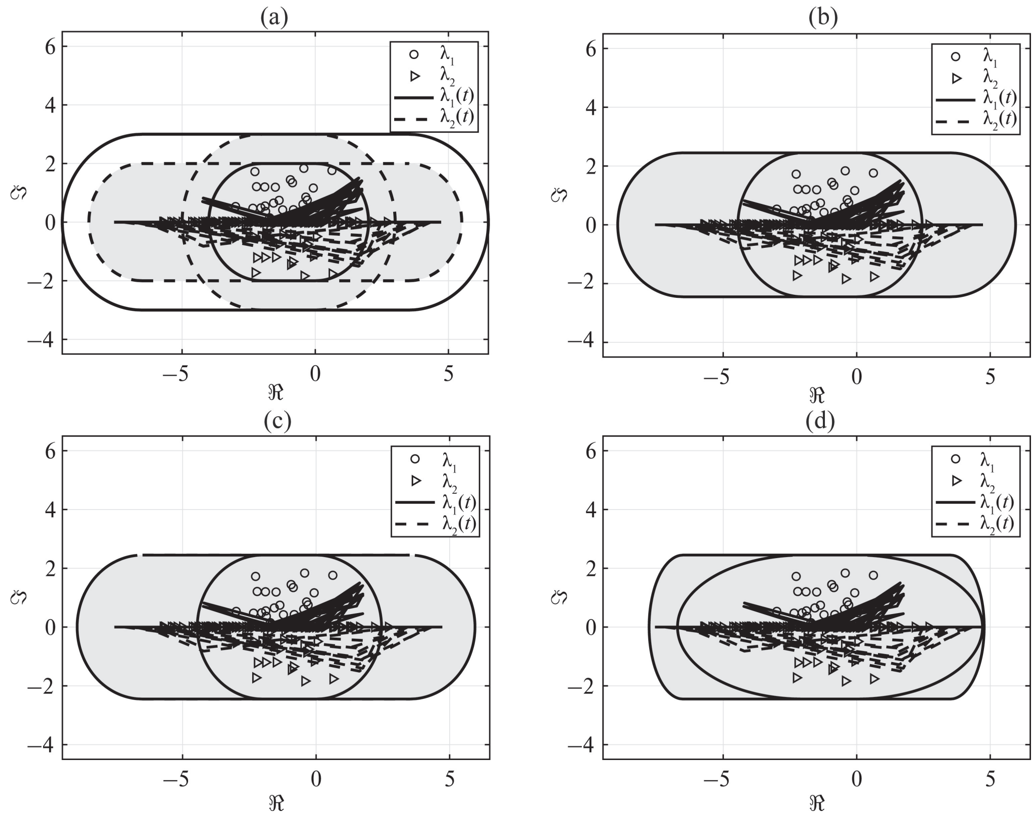

Fig. 2. Localization regions (gray region) of eigenvalues of perturbed matrices Q and $Q(t)$.

Example 2. Consider two parametrically indefinite matrices Q with constant and variable parameters in the forms

$$Q = \begin{bmatrix} -1 & 0 \\ 0 & -1.5 \end{bmatrix} + \underbrace{\begin{bmatrix} r_{11} & 2r_{12} \\ 3r_{21} & 4r_{22} \end{bmatrix}}_{\Delta Q},$$

$$Q(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1.5 \end{bmatrix} + \underbrace{\begin{bmatrix} \sin(t) & 2\cos(1.5t) \\ 3\mathrm{sgn}(\sin(2t)) & 4\mathrm{sgn}(\cos(1.7t)) \end{bmatrix}}_{\Delta Q(t)},$$

where r_{ij}, $i, j = 1, 2$ are pseudorandom numbers uniformly distributed over the interval $(-1; 1)$. Let us consider 100 realizations for each r_{ij}. The matrices ΔQ and $\Delta Q(t)$ have the same m_{ij}, so the estimates of the localization region will be the same.

Figure 2 shows the localization region of the eigenvalues of Q and $Q(t)$ using the results of Lemmas 3–5 (gray regions), where small circles and triangles represent the eigenvalues of the matrix Q with constant parameters, and continuous and dashed lines (inside the gray regions) represent the eigenvalues of the matrix Q with non-stationary parameters. In three out of four figures, the pairs of e-circles coincided due to the variation of d_i, h_i, α, and β, so only two e-circles are shown in three figures. The corresponding figures were obtained using:

- Lemma 3 (Fig. 2a);
- Lemma 4 with $D = diag\{1; 1.23\}$ and $H = diag\{1; 0.81\}$ (Fig. 2b);
- Lemma 5 with $\alpha = \beta = 0.5$ and $D = H = I$ (Fig. 2c);
- Lemma 5 with $\alpha = 0.5$ and $D = diag\{1; 0.52\}$ (Fig. 2d).

The solid boundary of the e-circles corresponds to the figures composed along the rows of the matrix, and the dotted boundary corresponds to the figures composed along the columns of the matrix.

3. CONTROL SYSTEM STABILITY ANALYSIS

This section will consider several applications of the results of the previous section to the analysis and design of control systems.

3.1. Synchronization of Network Systems

Consider a network system consisting of n interconnected agents of the form

$$\dot{x}_i = \sum_{j=1}^{n} q_{ij} x_j + u_i, \quad i = 1, \ldots, n, \tag{13}$$

where $x_i \in \mathbb{R}$, $u_i \in \mathbb{R}$ is the control signal, $|q_{ij}| \leqslant m_{ij}$. It is required to ensure that the condition $\lim\limits_{t \to \infty} x_i(t) = 0$ is satisfied for all x_i by choosing u_i, $i = 1, \ldots, n$.

Let us define the control laws

$$u_i = -q x_i, \quad i = 1, \ldots, n, \tag{14}$$

where $q > 0$.

Use the following notations: $x = col\{x_1, \ldots, x_n\}$, $Q_0 = -qI$, $\Delta Q = (q_{ij})$ and $Q = Q_0 + \Delta Q$. Then (13) and (14) can be rewritten as

$$\dot{x} = Qx. \tag{15}$$

As a result, checking the condition $\lim\limits_{t \to \infty} x_i(t) = 0$ is reduced to checking the stability of the matrix $Q_0 + \Delta Q$, which can be ensured by an appropriate choice of q in (14).

Let $q = -10^3$ and q_{ij} be pseudorandom numbers uniformly distributed over the interval $(-1; 1)$.

To analyze the stability of the matrix $Q_0 + \Delta Q$, we use:

- functions eig (calculating the eigenvalues of a matrix) and lyap (solving the Lyapunov equation) in MatLab, assuming that q_{ij} are known;
- applications to solving the linear matrix inequalities CVX and Yalmip/SeDuMi, assuming q_{ij} to be known;
- Lemma 1 (calculating $\sigma_{\max}\{Q\}$) and Lemma 2 (calculating $\sigma^{\alpha}_{\max}\{Q\}$ with an exhaustive search of α from 0 to 1 with a step of 0.1), assuming q_{ij} to be known;
- Corollary 3 (calculating $\sigma^{\alpha}_{\max}\{Q\}$ with an exhaustive search of α from 0 to 1 with a step of 0.1), assuming q_{ij} to be unknown, but with known m_{ij}.

Figure 3 shows the graphs of the time spent on the operation to determine the stability of $Q_0 + \Delta Q$ depending on the dimension of the matrix (the number of agents in the network) and using the corresponding method. Regardless of whether the corresponding method indicated that the matrix Q is stable or unstable, the corresponding time was recorded to clarify this issue. The calculations were performed in Matlab R2021b on a PC with an AMD Ryzen 5 PRO 4650U processor with Radeon Graphics 2.10 GHz and 8 GB of RAM. The results for CVX and Yalmip/SeDuMi, as well as for Lemma 2 and Corollary 3 were almost identical, so their graphs in Fig. 3 matched in pairs. We also note that when analyzing the proposed results, the maximum calculation time was not reached due to the fact that Matlab R2021b did not generate a matrix with a dimension greater than 25 000.

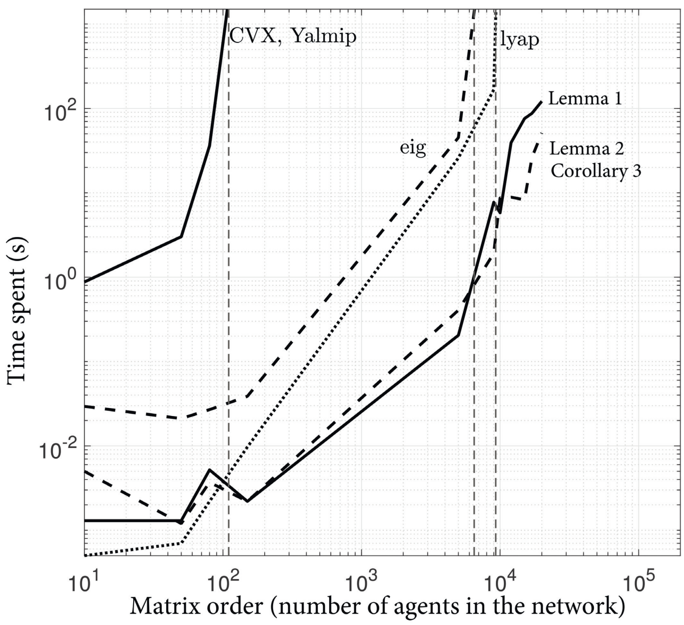

Fig. 3. Dependence of time spent on determining the stability of the system (matrix Q) on the number of agents in the network (n).

Conclusions:

- eig, lyap, CVX and Yalmip/SeDuMi algorithms provide a more accurate result in determining stability (they provide a smaller error in the deviation of the obtained solution from the true value) compared to the proposed estimates;
- the time spent on clarifying the stability issue when using CVX and Yalmip/SeDuMi increases significantly (to a lesser extent when using eig and lyap) with an increase in the number of agents in the network, while the proposed results are the least time-consuming in terms of calculation.

The closed-loop system (15) contains a matrix with diagonal dominance. In [6–18], where matrices with diagonal dominance were also used, it was noted that this is a rather narrow class of systems under study. In the following sections, we will show that the proposed results can be applied to systems with matrices without diagonal dominance. Diagonal dominance will be presented to expressions obtained using the apparatus of Lyapunov functions.

3.2. Stability Analysis of Linear Non-Stationary Systems with Interval-Uncertain Parameters and Matrices without Diagonal Dominance

In this section, we will consider a modification of the Demidovich theorem [19, Theorem 6.1; 23] (the term "Demidovich condition" is also used in the literature) on the study of the stability of linear systems with known non-stationary parameters in the case of interval uncertainty and the presence of external disturbances. Let the system be represented by the equation

$$\dot{x}(t) = A(t)x(t) + F(t)f(t), \tag{16}$$

where $t \geqslant 0$, $x \in \mathbb{R}^n$ is the state vector, $f \in \mathbb{R}^l$ is an external signal such that $\sup\{|f(t)|\} \leqslant \bar{f}$, $F(t) \in \mathbb{R}^{n\times l}$ and $A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}$ are such that $\sup\{\|F(t)\|\} \leqslant \bar{F}$, $A(t) = A_0 + \Delta A(t)$, $A_0 = (a^0_{ij})$, $\Delta A(t) = (\Delta a_{ij}(t))$, $\Delta \underline{a}_{ii} \leqslant \Delta a_{ii}(t) \leqslant \Delta \overline{a}_{ii}$ and $|\Delta a_{ij}(t)| \leqslant m_{ij}$ for $i \neq j$ and for all t.

Let us introduce the matrix $\bar{A}(t)$, where

$$\begin{gathered}\bar{A}(t) = A(t) + A^{\mathrm{T}}(t) = \bar{A}_0 + \Delta\bar{A}(t),\\ \bar{A}_0 = (\bar{a}_{ij}^0) = (a_{ij}^0 + a_{ji}^0),\\ \Delta\bar{A}(t) = (\Delta\bar{a}_{ij}(t)) = (\Delta a_{ij}(t) + \Delta a_{ji}(t)),\\ 2\Delta\underline{a}_{ii} \leqslant \Delta\bar{a}_{ii}(t) \leqslant 2\Delta\overline{a}_{ii},\\ |\Delta\bar{a}_{ij}(t)| \leqslant m_{ij} + m_{ji} \quad \text{at} \quad i \neq j.\end{gathered} \tag{17}$$

Note that the system (16) contains a matrix $A(t)$ without diagonal dominance. As will be shown in the theorem below, diagonal dominance will be needed in the matrix $\bar{A}(t)$.

According to Demidovich theorem [19, Theorem 6.1; 23], the system (16) is asymptotically stable for $f(t) \equiv 0$ and with a known matrix $A(t)$ if the eigenvalues of the matrix $A(t) + A^{\mathrm{T}}(t)$ take negative values for all t. Next, we consider a generalization of this theorem to interval indefinite matrices, taking into account the Corollary 3.

Theorem 1. *Denote by σ any upper bound calculated using* (12) *for the eigenvalues of the matrix $\bar{A}(t)$ in* (17). *If $\sigma < 0$, then the following bound holds:*

$$|x(t)| \leqslant -\frac{2\|\bar{F}\|\bar{f}}{\sigma} + \mathcal{C}e^{0.5\sigma t}, \tag{18}$$

where $\mathcal{C} = \max\left\{0, |x(0)| + \frac{2\|\bar{F}\|\bar{f}}{\sigma}\right\}$.

Proof. We choose the Lyapunov function

$$V = x^{\mathrm{T}}x \tag{19}$$

and find its derivative along the solutions (16) in the form

$$\dot{V} = x^{\mathrm{T}}\bar{A}(t)x + 2x^{\mathrm{T}}F(t)f.$$

Let us find the upper estimate

$$\dot{V} \leqslant \sigma x^{\mathrm{T}}x + 2|x|\|F(t)\||f| \leqslant \sigma V + 2\sqrt{V}\|\bar{F}\|\bar{f}. \tag{20}$$

We solve the inequality (20) in the form

$$\sqrt{V} \leqslant -\frac{2\|\bar{F}\|\bar{f}}{\sigma} + \left(\sqrt{V(0)} + \frac{2\|\bar{F}\|\bar{f}}{\sigma}\right)e^{0.5\sigma t}. \tag{21}$$

Taking into account (19), we obtain

$$|x(t)| \leqslant -\frac{2\|\bar{F}\|\bar{f}}{\sigma} + \left(|x(0)| + \frac{2\|\bar{F}\|\bar{f}}{\sigma}\right)e^{0.5\sigma t}. \tag{22}$$

The expression (22) yields the result (18).

Example 3. A system with constant parameters and a matrix without diagonal dominance. Consider the system (16) with parameters $A = \begin{bmatrix} -1 & 3 \\ -2.5 & -2 \end{bmatrix}$, $B = [0\ 0.05]^{\mathrm{T}}$ and $u = \sin(t)$. The matrix A is not superstable [6–8] or diagonally dominant [4, 9–18] either in rows or in columns. There are also no $d_1 > 0$ and $d_2 > 0$ for the conditions (2) to be satisfied, since the inequalities

$d_1 - 3d_2 > 0$ and $-2.5d_1 + 2d_2 > 0$, composed for the matrix A, and the inequalities $d_1 - 2.5d_2 > 0$ and $-3d_1 + 2d_2 > 0$, composed for the matrix A^{T}, have no solution.

Consider the matrix $\bar{A} = A + A^{\mathrm{T}} = \begin{bmatrix} -2 & 0.5 \\ 0.5 & -4 \end{bmatrix}$. The condition (2) will be satisfied for $\bar{A}$, where $\sigma = \sigma_{\max}(\bar{A}) = -1.5$. The largest eigenvalue of the matrix $A + A^{\mathrm{T}}$ is -1.88. If we use another condition in (2) with $d_1 = 1$ and $d_2 = 0.711$, then the eigenvalue estimate can be improved to $\sigma = \sigma^D_{\max}(\bar{A}) = -1.6445$.

Example 4. A system with non-stationary parameters with a matrix without diagonal dominance. Consider the system (16) with parameters with $A(t) = A_0 + \Delta A(t)$, where $A_0 = A$ from the previous example, $\Delta A(t) = 0.1 \begin{bmatrix} \sin(t) & \cos(t) \\ \sin(2t) & \sin(4t) \end{bmatrix}$. The upper bounds (17) give negative values, therefore, the system (16) is stable.

3.3. Control Law Design for Linear Systems with Matrices without Diagonal Dominance

Consider the system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + F(t)f(t), \tag{23}$$

where $u \in \mathbb{R}^m$ is the control signal, $B(t) \in \mathbb{R}^{n\times m}$, $B(t) = b(t)B_0$, $\underline{b} \leqslant b(t) \leqslant \overline{b} \in \mathbb{R}$, B_0 is a known matrix, the pair $(A(t), B(t))$ is controllable for all t. The remaining notations are as in (16). Assume that the parameters $\Delta A(t)$, $b(t)$, $F(t)$ and $f(t)$ are unknown.

Introduce the control law

$$u = Kx, \tag{24}$$

where $K \in \mathbb{R}^{m\times n}$. Below we formulate theorems that allow us to calculate the matrix K that ensures the stability of the closed-loop system

$$\dot{x}(t) = (A(t) + B(t)K)x(t) + F(t)f(t). \tag{25}$$

Note that neither the matrix $A(t)$ nor the matrix $A(t)+B(t)K$ requires the diagonal dominance property to be satisfied.

Theorem 2. *Let the matrices A, B, and F in* (23) *be known and constant, and let for the given $\alpha > 0$ there exist the matrix $Q = Q^{\mathrm{T}}$ and the coefficient $\beta > 0$ such that the following conditions are satisfied:*

$$\begin{gathered} \Psi_{ii} < 0, \\ \Psi_{ij} \geqslant 0 \textit{ for } i \neq j,\ i,j = 1,\ldots,n, \\ \sigma(Q) > 0, \end{gathered} \tag{26}$$

where

$$\Psi = (\Psi_{ij}) := QA^{\mathrm{T}} + AQ + Y^{\mathrm{T}}B^{\mathrm{T}} + BY + \alpha Q + \beta F^{\mathrm{T}}F, \tag{27}$$

$\sigma(Q)$ is one of the lower bounds for the eigenvalues of matrix Q, obtained using (2), *as well as $K = YQ^{-1}$ and $P = Q^{-1}$. Then for the solutions of the system* (25) *the following estimate will be valid*

$$|x(t)| \leqslant \left[\frac{1}{\lambda_{\min}(P)}\left(\frac{\bar{f}^2}{\alpha\beta} + \mathcal{K}e^{-\alpha t}\right)\right]^{0.5}, \tag{28}$$

where $\mathcal{K} = \max\left\{0, x(0)^{\mathrm{T}}Px(0) - \frac{\bar{f}^2}{\alpha\beta}\right\}$.

Proof. We choose the Lyapunov function

$$V = x^{\mathrm{T}} P x, \tag{29}$$

where $P = Q^{-1}$, and find its time derivative along the solutions (25):

$$\dot{V} = x^{\mathrm{T}}[(A + BK)^{\mathrm{T}} P + P(A + BK)]x + 2x^{\mathrm{T}} F f. \tag{30}$$

Denoting $z = col\{x, f\}$ and substituting (29) and (30) into the exponential stability condition $\dot{V} + \alpha V + \gamma f^{\mathrm{T}} f < 0$, $\gamma = 1/\beta$, we obtain

$$z^{\mathrm{T}} \begin{bmatrix} (A + BK)^{\mathrm{T}} P + P(A + BK) + \alpha P & PF \\ \star & -\gamma I \end{bmatrix} z < 0. \tag{31}$$

According to [24], the inequality (31) will be satisfied if the following condition is satisfied:

$$\begin{bmatrix} (A + BK)^{\mathrm{T}} P + P(A + BK) + \alpha P & PF \\ \star & -\gamma I \end{bmatrix} < 0. \tag{32}$$

Using Schur lemma [24] and the fact that $\gamma = 1/\beta$, we rewrite (32) as

$$(A + BK)^{\mathrm{T}} P + P(A + BK) + \alpha P + \beta P F^{\mathrm{T}} F P < 0. \tag{33}$$

Multiplying (33) on the left and right by Q^{-1} and replacing $Y = KQ$, we get

$$\Psi := QA^{\mathrm{T}} + AQ + Y^{\mathrm{T}} B^{\mathrm{T}} + BY + \alpha Q + \beta F^{\mathrm{T}} F < 0. \tag{34}$$

According to Lemmas 1 and 2, the eigenvalues of the symmetric matrices Ψ and Q will be negative and positive, respectively, if the inequalities (26) are satisfied. On the other hand, according to [4] (Theorem 7.2.1), a Hermitian matrix is positive (negative) definite if and only if all its eigenvalues are positive (negative). Therefore, the conditions $\Psi < 0$ and $Q > 0$ will be satisfied if the inequalities (26) are satisfied. The estimate (28) follows from the solution of the inequality $\dot{V} + \alpha V + \gamma f^{\mathrm{T}} f < 0$ taking into account (29) and the estimate $\lambda_{\min}\{P\}|x|^2 \leqslant x^{\mathrm{T}} P x$.

Using the results of Theorem 2, we formulate the following theorem for systems with unknown non-stationary parameters.

Theorem 3. *Consider the system* (23) *with non-stationary parameters. Let there exist the matrix* $Q = Q^{\mathrm{T}}$ *and the coefficient* $\beta > 0$ *such that the conditions hold*

$$\begin{gathered} \Phi_{ii} < 0, \\ \Phi_{ij} \geqslant 0 \text{ for } i \neq j, \\ \sigma(Q) > 0, \end{gathered} \tag{35}$$

at the vertices $|\Delta a_{ij}(t)| \leqslant m_{ij}$ *and* $\underline{b} \leqslant b(t) \leqslant \overline{b}$, *where*

$$\Phi = (\Phi_{ij}) := QA_0^{\mathrm{T}} + A_0 Q + Q\Delta A^{\mathrm{T}}(t) + \Delta A(t) Q + b(t) Y^{\mathrm{T}} B_0^{\mathrm{T}} + b(t) B_0 Y + \alpha Q + \beta \bar{F}^2 I,$$

$\sigma(\Psi)$ *is one of the upper bounds of the matrix* Ψ, *obtained using* (12), *and also* $K = YQ^{-1}$ *and* $P = Q^{-1}$. *Then for solutions of the system* (25) *the estimate* (28) *will be valid.*

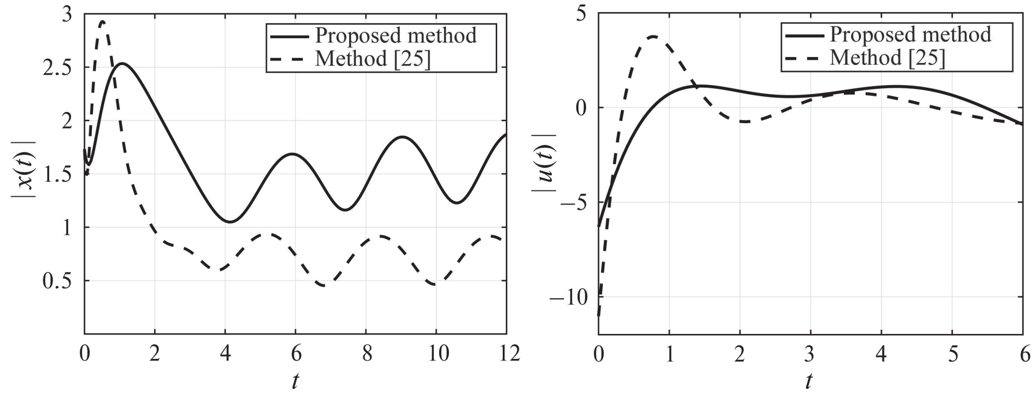

Fig. 4. The transients of $|x(t)|$ and $u(t)$ for the proposed algorithm (solid curves) and the algorithm [25] (dashed curves).

Proof. We will use the results (29)–(34) from the proof of Theorem 2 taking into account non-stationary parameters. Since $A(t) = A_0 + \Delta A(t)$, $B(t) = b(t)B_0$ and $\|F(t)\| \leqslant \bar{F}$, then we rewrite (34) as

$$\Phi := QA_0^{\mathrm{T}} + A_0Q + Q\Delta A^{\mathrm{T}}(t) + \Delta A(t)Q + b(t)Y^{\mathrm{T}}B_0^{\mathrm{T}} + b(t)B_0Y + \alpha Q + \beta\bar{F}^2 I < 0.$$

If the conditions (35) are satisfied at the vertices $|\Delta a_{ij}(t)| \leqslant m_{ij}$ and $\underline{b} \leqslant b(t) \leqslant \overline{b}$, then according to [24] the condition (35) will be satisfied for any $\Delta A(t)$ and $b(t)$ inside the polytope with vertices $|\Delta a_{ij}(t)| \leqslant m_{ij}$ and $\underline{b} \leqslant b(t) \leqslant \overline{b}$. The estimate (28) is obtained similarly to the proof of Theorem 2.

Example 5. Consider the system (23) with parameters $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, $B = col\{0; 0; 1\}$, $F = col\{0.1; 0.5; 1\}$ and $f(t) = \sin(t)$.

Obviously, the matrix A is without diagonal dominance, and the structure of the matrix B does not allow the control law $u = Kx$ with $K \in \mathbb{R}^{1\times 3}$ to lead to the closed-loop system with a matrix with diagonal dominance. Therefore, we use Theorem 2 to analyze the localization region of the eigenvalues of the matrix Φ obtained as a result of applying the Lyapunov function method. Using Theorem 2, we obtain $K = col\{-1.3671; -2.3619; -2.5724\}$ and $trace(P) = 25.5858$ for $\alpha = 1$. Using [25], we obtain $K = col\{-2.8862; -4.9244; -3.2136\}$ and $trace(P) = 40.631$ for $\alpha = 1$. In both cases, the goal was $trace(P) \to \min$ for calculating K.

From Fig. 4 it is evident that in the steady state the value of $|x(t)|$ of the proposed algorithm is greater. However, the spike of $|x(t)|$ and the amplitude of the control signal $u(t)$ at the initial moment of time are smaller, and the value of $trace(P)$ is also smaller.

4. CONCLUSION

The paper has described the application of the Gershgorin theorem and theorems derived from it for estimating the localization domain of the eigenvalues of a matrix with constant and known parameters. These results are generalized to estimate the localization domain for matrices with parametric interval uncertainty. The concept of an e-circle is proposed, which allows obtaining more accurate estimates of the localization domain than a direct application of the Gershgorin theorem. The obtained results are applied to the control of network systems, where it is shown that for large-dimensional problems, the proposed results are the least time-consuming in terms of execution time compared to the eig and lyap procedures (commands in MatLab for finding the eigenvalues of a matrix and solving the Lyapunov equation), as well as CVX and Yalmip/SeDuMi for solving linear matrix inequalities. A generalization of the Demidovich condition is proposed for determining the stability of a non-stationary matrix. An approach has been developed for calculating the matrix in a linear control law for control of linear systems where the property of diagonal dominance for matrices in the closed-loop system is not fulfilled.

FUNDING

The work was carried out at the Institute for Problems in Mechanical Engineering of the Russian Academy of Sciences with the support of Goszadanie no. FFNF-2024-0008 (no. 124041100006-1 in EGISU NIOKTR).

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This paper was recommended for publication by M.V. Khlebnikov, a member of the Editorial Board